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LETTER TO THE EDITOR

Supersymmetry on Jacobstahl lattices

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Abstract

It is shown that the construction of Yang and Fendley (2004 *J. Phys. A: Math. Gen.* **37** 8937) to obtain supersymmetric systems leads not to the open XXZ chain with anisotropy $\Delta = -\frac{1}{2}$ but to systems having dimensions given by Jacobstahl sequences. For each system the ground state is unique. The continuum limit of the spectra of the Jacobstahl systems coincide, up to degeneracies, with that of the $U_q(sl(2))$ invariant XXZ chain for $q = \exp(\mathrm{i}\pi/3)$. The relation between the Jacobstahl systems and the open XXZ chain is explained.

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Yang and Fendley [1] have given a construction to obtain supersymmetric systems given by Hamiltonians H obeying $N = 2$ supersymmetry:

$$\{Q, Q^\dagger\} = H, \qquad Q^2 = 0, \tag{1}$$

$$[F, Q] = -Q. \tag{2}$$

Here Q and Q^+ are the supercharges and F is the Fermion number operator. Their results imply that a special combination

$$H = \bigoplus_{L=1}^{\infty} H_{XXZ}^{(L)} \tag{3}$$

of the L site XXZ open quantum chains, at anisotropy $\Delta = -1/2$, which in terms of the standard Pauli matrices are given by

$$H_{XXZ}^{(L)} = -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \frac{1}{2}\sigma_i^z\sigma_{i+1}^z\right) - \frac{1}{4}\left(\sigma_1^z + \sigma_L^z\right) + \frac{3L-1}{4} \tag{4}$$

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is supersymmetric. They also show that the ground-state energy of the Hamiltonian given by (4) is zero. The last fact is correct but can be proven by other methods [2]. However for the Hamiltonian (3) one cannot derive a continuum limit or even compare it with experiments. In this letter we are going to show that another use of the supercharges defined by Yang and Fendley [1] can lead to the construction of finite-dimension quantum spin systems with a well-defined thermodynamical limit ($L \to \infty$). We also give a possible explanation of the observation [3] that the ground-state wavefunction of $H_{XXZ}^{(L)}$ is given by positive integers.

Instead of defining the Q operator given in (1)–(2) in terms of fermionic operators, as done in [1] we give their matrix elements on an appropriate representation of the vector space. This operator, that we denote by $Q^{(L,L+1)}$, acts on a Hilbert space of dimension $2^L \oplus 2^{L+1}$ spanned by the vector basis $\{|v_L\rangle\} \oplus \{|v_{L+1}\rangle\}$, where $\{|v_L\rangle\} = \{|s_1,\ldots,s_L\rangle\}, \{|v_{L+1}\rangle\} = \{|s'_1,\ldots,s'_{L+1}\rangle\}$, $s_i, s'_j = \pm$, is the standard σ_z- basis, namely,

$$
\begin{aligned}
&Q^{(L,L+1)} = \sum_{j=1}^{L} Q_j^{(L,L+1)}, \\
&Q_j^{(L,L+1)}|s'_1,\ldots,s'_j,\ldots,s'_{L+1}\rangle = 0, \\
&Q_j^{(L,L+1)}|s_1,\ldots,s_j,\ldots,s_L\rangle = (-)^{j-i}|s_1,\ldots,s_{j-1},+,+,s_{j+1},\ldots,s_L\rangle\delta_{s_j,-},
\end{aligned}
\tag{5}
$$

where $(j = 1,\ldots,L)$. It is immediate from (5) to see that $Q_i^{(L,L+1)}Q_j^{(L,L+1)} = 0$, implying $(Q^{(L,L+1)})^2 = 0$. We can define non-local L-site quantum chains $H_1^{(L)}$ and $H_2^{(L)}$ acting on the vector space spanned by $\{|v_L\rangle\}$ of dimension 2^L by

$$
\begin{aligned}
Q^{(L,L+1)}Q^{(L,L+1)\dagger} &= O^{(L)} \oplus H_2^{(L+1)}, \\
Q^{(L,L+1)\dagger}Q^{(L,L+1)} &= H_1^{(L)} \oplus O^{(L+1)},
\end{aligned}
\tag{6}
$$

where $O^{L'}$ $(L' = L, L+1)$ is the matrix with zero elements in the space of dimension $2^{L'}$ spanned by $\{|v_{L'}\rangle\}$. We can verify the following properties of $H_1^{(L)}$ and $H_2^{(L)}$:

$$
\begin{aligned}
&H_1^{(L)\dagger} = H_1^{(L)}, \qquad H_2^{(L)\dagger} = H_2^{(L)}, \qquad \left[S^z, H_1^{(L)}\right] = \left[S^z, H_2^{(L)}\right] = 0, \\
&S^z|s_1,\ldots,s_L\rangle = \left(\sum_{i=1}^{L} s_i\right)|s_1,\ldots,s_L\rangle.
\end{aligned}
\tag{7}
$$

We can also verify that

$$
H_s^{(L+1)} = \left\{Q^{(L,L+1)}, Q^{(L,L+1)\dagger}\right\} = H_1^{(L)} \oplus H_2^{(L+1)} \neq H_{XXZ}^{(L)} \tag{8}
$$

and consequently the L site XXZ open chain is not supersymmetric.

The correct relation among these quantum chains with the $H_{XXZ}^{(L)}$ is given by

$$
H_{XXZ}^{(L)} = H_1^{(L)} + H_2^{(L)}. \tag{9}
$$

Moreover we can also verify that

$$
H_1^{(L)}H_2^{(L)} = H_2^{(L)}H_1^{(L)} = 0, \tag{10}
$$

which imply that $H_1^{(L)}, H_2^{(L)}$ and $H_{XXZ}^{(L)}$ share the same eigenvectors, and the non-zero eigenvalues of $H_1^{(L)}$ or $H_2^{(L)}$ are the same as those of $H_{XXZ}^{(L)}$. For general values of L

the Hamiltonian $H_1^{(L)}$ obtained from (6) is given by

$$
\begin{aligned}
H_1^{(L)} = &-\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^z - 1\right) - \sum_{i=1}^{L-1}\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+\right) \\
&+\frac{1}{8}\sum_{i=1}^{L-2}\sum_{k=i+2}^{L}(-)^{k-i}\sigma_i^-\sigma_k^+ A_{i,k}\left(\sigma_i^z+1\right)\left(\sigma_{i+1}^z+1\right)\left(1-\sigma_k^z\right) \\
&+\frac{1}{8}\sum_{i=3}^{L}\sum_{k=1}^{i-2}(-)^{k-i}\sigma_i^-\sigma_k^+ A_{k,i}^\dagger\left(\sigma_{i-1}^z+1\right)\left(\sigma_i^z+1\right)\left(1-\sigma_k^z\right),
\end{aligned}
\qquad (11)
$$

where

$$
A_{i,k} = \begin{cases} 0 & \text{if} \quad k < i+2 \\ 1 & \text{if} \quad k = i+2 \\ \left(\frac{1}{2}\vec{\sigma}_{k-2}\vec{\sigma}_{k-1}+\frac{1}{2}\right)\cdots\left(\frac{1}{2}\vec{\sigma}_{i+2}\vec{\sigma}_{i+3}+\frac{1}{2}\right)\left(\frac{1}{2}\vec{\sigma}_{i+1}\vec{\sigma}_{i+2}+\frac{1}{2}\right) & \text{if} \quad k > i+2 \end{cases}
$$

and $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$, $\sigma^\pm = (\sigma^x \pm \sigma^y)/2$ and $[L/2] = \mathrm{Int}(L/2)$. The quantum chain $H_2^{(L)}$ is obtained from $H_1^{(L)}$ and $H_{XXZ}^{(L)}$ by using (4) and (9) . Some examples for $L = 2$ and $L = 3$ are

$$
\begin{aligned}
H_1^{(2)} &= -\tfrac{1}{2}\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y\right) - \tfrac{1}{2}\left(\sigma_1^z+\sigma_2^z\right) + 1, \\
H_2^{(2)} &= \tfrac{1}{4}\sigma_1^z\sigma_2^z + \tfrac{1}{4}\left(\sigma_1^z+\sigma_2^z\right) + \tfrac{1}{4},
\end{aligned}
\qquad (12)
$$

$$
\begin{aligned}
H_1^{(3)} &= -\tfrac{1}{2}\Big[\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_2^x\sigma_3^x + \sigma_2^y\sigma_3^y\right) - \tfrac{1}{2}\left(\sigma_1^x\sigma_3^x + \sigma_1^y\sigma_3^y\right)\left(1+\sigma_2^z\right) \\
&\qquad + \sigma_1^z + \sigma_2^z + \sigma_3^z - 3\Big], \\
H_2^{(3)} &= -\tfrac{1}{4}\Big[\left(\sigma_1^x\sigma_3^x + \sigma_1^y\sigma_3^y\right)\left(1+\sigma_2^z\right) - \left(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z\right) - \left(\sigma_1^z + 2\sigma_2^z + \sigma_3^z\right) - 2\Big].
\end{aligned}
\qquad (13)
$$

Note that in order to find the eigenvalues and eigenfunctions of $H_1^{(L)}$ and $H_2^{(L)}$ one has to use the eigenfunctions of $H_{XXZ}^{(L)}$ which can be obtained using the Bethe ansatz. Of the 2^L eigenvalues of $H_{XXZ}^{(L)}$,

$$
\frac{1}{3}\left[2^L - \frac{[3-(-1)^L]}{2}\right] \qquad (14)
$$

can be found in $H_2^{(L)}$ and the remaining

$$
\frac{1}{3}\left[2^{L+1} + \frac{[3-(-1)^L]}{2}\right] \qquad (15)
$$

in $H_1^{(L)}$. The ground-state energy being included in this last set. All the eigenvalues of $H_1^{(L)}$ and $H_2^{(L)}$ not belonging to the sets (14) and (15) are equal to zero.

The quantum chain $H_{XXZ}^{(L)}$ although having a zero-energy ground state and all the eigenlevels real and positive numbers is not supersymmetric. The Hamiltonian $H_s^{(L+1)}$ with 3×2^L states given by (8) is supersymmetric. The supercharges connect states with $S^z = m$ in the 2^L vector space with states with $S^z = m + 3$ in the $2^{(L+1)}$ vector space. The non-zero energies appear in doublets (some of them degenerate) and the zero energy is highly degenerate. The degeneracies of the zero energy level can be computed using equations (14) and (15). Can we define supersymmetric systems with an unique ground state and the other energy levels coinciding with those of $H_s^{(L)}$? The answer is yes but the path is long. One starts with $L = 2$ and takes the two states corresponding to $S^z = 0$ ($|-+\rangle$ and $|+-\rangle$) and one state with $S^z = -2$ ($|--\rangle$) to which we apply $Q^{(2,3)}$ defined by (5). One gets the two states $|+++\rangle$ and ($|++-\rangle - |-++\rangle$). Using this definition of $Q^{(2,3)}$ in this subspace only, one obtains a supersymmetric system with five states defined by the Hamiltonian $H_J^{(3)}$. What we have done is to truncate the vector space of $H_1^{(2)} \oplus H_2^{(3)}$. We denote by $H_{1,t}^{(2)}$ and $H_{2,t}^{(3)}$ the Hamiltonians acting in the truncated spaces. Obviously

$$H_J^{(3)} = H_{1,t}^{(2)} \oplus H_{2,t}^{(3)}. \tag{16}$$

In order to obtain the truncated vector space in which $H_{1,t}^{(3)}$ acts, one has to take the vector space which is orthogonal to the one which $H_{2,t}^{(3)}$ acted. This is a six-dimensional vector space: $S^z = -3$ (1 state), $S^z = -1$ (3 states) and $S^z = 1$ (2 states: ($|++-\rangle + |-++\rangle$) and $|+-+\rangle$). One can proceed further. (A look at appendix A of [5] might help the reader to follow the steps). One uses $Q^{(3,4)}$ applied to the six states to find the vector space in which $H_{2,t}^{(4)}$ acts and find that supersymmetric Hamiltonian $H_J^{(4)}$ which acts in an 11-dimensional space etc . . . Applying consistently this procedure, we find that the supersymmetric Hamiltonian

$$H_J^{(L)} = H_{1,t}^{(L-1)} \oplus H_{2,t}^{(L)} \tag{17}$$

acts in a vector space of dimension

$$\tfrac{1}{3}[2^{L+1} + (-1)^L]. \tag{18}$$

The numbers obtained using equation (18) are called Jacobstahl numbers and they have interesting combinatorial interpretations [4]. One can use now the results of [5] (equations (3.5) and (3.6)) to obtain the spectrum of $H_J^{(L)}$. It can be obtained from the spectrum of an $U_q(sl(2))$ (for $q = e^{i\pi/3}$) symmetric quantum chain with L sites (see [5], equation (2.12)) using the following rule: (a) there is a unique ground state of energy zero (if L is odd, the ground state of the $U_q(sl(2))$ symmetric chain is doubly degenerate) (b) the degeneracy of a non-zero energy level in $H_J^{(L)}$ is equal to 2/3 the degeneracy of the same level in the $U_q(sl(2))$ symmetric chain. Using this observation and the known results on the finite-size scaling of the spectra of the $U_q(sl(2))$ symmetric chain [6] for $q = e^{i\pi/3}$ one can easily derive the conformal properties of the Jacobstahl systems.

We did not try to generalize our observations to other spin chains.

Before closing our letter, let us mention a fact observed independently by several people [3]. The ground-state wavefunction of $H_{XXZ}^{(L)}$ given by equation (4) has positive integer coefficients. Positive coefficients occur if the ground-state wavefunction is associated with a Hamiltonian which describes a stochastic process (the ground-state energy has to be zero, which is the case). The ground-state wavefunction of a stochastic process can be interpreted as a probability distribution function and therefore has positive coefficients. One can show that in $H_J^{(L)}$, in the sector in which the ground state is found, one can define a stochastic process. If the fact that the coefficients are integer has a more profound explanation, remains to be seen.

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